ISRAEL JOURNAL OF MATHEMATICS **124** (2001), 1–27

VALUES OF NON-ATOMIC VECTOR MEASURE GAMES

BY

ABRAHAM NEYMAN*

Institute of Mathematics and Center for Rationality
The Hebrew University of Jerusalem, Givat Ram, Jerusalem 91904, Israel
e-mail: aneyman@math.huji.ac.il

ABSTRACT

There is a value (of norm one) on the closed space of games that is generated by all games of bounded variation $f \circ \mu$, where μ is a vector of non-atomic probability measures and f is continuous at $0 = \mu(\emptyset)$ and at $\mu(I)$.

1. Introduction

The Shapley value is one of the basic solution concepts of cooperative game theory. It can be viewed as a sort of average or expected outcome, or as an a priori evaluation of the players' expected payoffs.

The value has a very wide range of applications in fields as diverse as economics and political science. In many of these applications it is necessary to consider games that involve a large number of players. Often, most of the players are individually insignificant, and are effective in the game only via coalitions. A typical example is a perfectly competitive market. At the same time there may exist big players who retain the power to wield single-handed influence. A typical example is provided by voting among stockholders of a corporation, with a few major stockholders and an "ocean" of minor stockholders. In economics, one considers an oligopolistic sector of firms embedded in a large population of "perfectly competitive" consumers. In all these cases, it is fruitful to model the game as one with a continuum of players. In general, the continuum consists of a non-atomic

* This research was supported by NSF Grant SES 8922610.
Received December 24, 1998

part (the "ocean"), along with (at most countably many) atoms. The continuum provides a convenient framework for mathematical analysis, and approximates the results for large finite games well. Also, it enables a unified view of games with finite or countable or oceanic player-sets, or indeed any mixture of these.

The space of players is represented by a measurable space $(I, \mathcal{C})$ that is isomorphic to $[0,1]$ with the Borel σ-field. The members of the set I are called **players**, those of $\mathcal{C}$, **coalitions.** A game is a real-valued function v on $\mathcal{C}$ such that $v(\emptyset) = 0$. For each coalition S in $\mathcal{C}$, the number $v(S)$ is interpreted as the total payoff that the coalition S, if it forms, can obtain for its members. A game v is **finitely additive** if $v(S \cup T) = v(S) + v(T)$ whenever S and T are two disjoint coalitions. A distribution of payoffs is represented by a finitely additive game. A value is a mapping from games to distributions of payoffs, i.e., to finitely additive games that satisfy several plausible conditions: linearity, symmetry, positivity and efficiency. There are several additional desirable conditions, e.g., continuity, strong positivity, a null player axiom, a dummy axiom, diagonality, and so on.

A game v is **monotonic** if $v(S) \geq v(T)$ whenever $S \supset T$. The **variation** of a game v, denoted $\|v\|$, is the supremum of the variation of v over all increasing chains $S_1 \subset S_2 \subset \cdots \subset S_n$ in $\mathcal{C}$. A game v has bounded variation if $\|v\| < \infty$. The space of all games of bounded variation, BV, is a Banach space w.r.t. the variation norm.

Denote by $\mathcal{G}$ the group of automorphisms (i.e., one-to-one measurable mappings θ from I onto I with θ^{-1} measurable) of the underlying space $(I, \mathcal{C})$. Each θ in $\mathcal{G}$ induces a linear mapping θ_* of BV onto itself, defined by $(\theta_* v)(S) = v(\theta S)$. A set of games Q is called **symmetric** if $\theta_* Q = Q$ for all θ in $\mathcal{G}$.

The space of all finitely additive and bounded games is denoted FA; the subspace of all measures (i.e., countably additive games) is denoted M; and its subspace consisting of all non-atomic measures is denoted NA. Obviously, $NA \subset M \subset FA \subset$ BV, and each of the spaces NA, M, FA, and BV is a symmetric space.

Given a set of games Q, we denote by Q^+ all monotonic games in Q, and by Q^1 the set of all games v in Q^+ with $v(I) = 1$. An operator $\varphi : Q \to BV$ is called **positive** if $\varphi(Q^+) \subset BV^+$; **symmetric** if for every $\theta \in \mathcal{G}$ and v in Q, $\theta_* v \in Q$ implies that $\varphi(\theta_* v) = \theta_*(\varphi v)$; and **efficient** if for every v in Q, $(\varphi v)(I) = v(I)$.

Let Q be a symmetric linear subspace of BV. A **value** on Q is a linear operator φ: $Q \to FA$ that is symmetric, positive, and efficient.

There are several spaces of games that have a value. One of them is the space of all games with a finite support; other spaces are pNA (pM, pFA)—the closed (in the bounded variation norm) algebra generated by NA (M, FA, respectively). Examples of games in pNA are games of the form $f \circ \mu$, where $\mu = (\mu_1, \ldots, \mu_n)$ is a vector of non-atomic probability measures and the function f is continuously differentiable on the range of μ. The value of such a game is given by the diagonal formula

$$\varphi(f \circ \mu)(S) = \int_0^1 f_{\mu(S)}(t\mu(I))dt$$

where $f_{\mu(S)}$ is the directional derivative of f in the direction $\mu(S)$.

Other spaces on which a value—of norm 1—is known to exist are $bv'NA$ ($bv'M$, $bv'FA$)—the closed linear space generated by games of the form $f \circ \mu$ where $f \in bv' =: \{f\colon [0,1] \to R \mid f$ is of bounded variation and continuous at 0 and at 1 with $f(0) = 0\}$, and $\mu \in NA^1$ ($\mu \in M^1, \mu \in FA^1$). Obviously, $pNA \subset bv'NA \subset bv'M \subset bv'FA$. Aumann and Shapley [1] proved the existence of a (unique) value on each of the spaces pNA and $bv'NA$. Existence of a value on $bv'M$ and $bv'FA$ is proved in Mertens [6] and Neyman [10].

Mertens [6] constructs a value—of norm 1—on a closed space $\mathcal{M}$ that includes the algebra generated by $bv'NA$, the space $bv'M$ and also all games generated by a finite number of algebraic and lattice operations from a finite number of measures, and all market functions of finitely many measures.

The present paper constructs a value on essentially all games that are functions of finitely many non-atomic measures. More explicitly, given a vector μ of non-atomic probability measures on the space of players $(I, \mathcal{C})$, we denote by $Q(\mu)$ the space of all games of bounded variation that are of the form $f \circ \mu$, where f is a real-valued function defined on the range of the vector measure $\mu = (\mu_1, \ldots, \mu_n)$ and continuous at $0 = \mu(\emptyset)$ and $\mu(I)$. The space Q is the union of all spaces $Q(\mu)$ where μ ranges over all vectors of non-atomic probability measures. Our result shows that there is a value—of norm 1—on Q and thus also on its closure:

THEOREM 1: (i) *There is a value of norm 1 on the closed subspace of BV that is generated by all games of the form $f \circ \mu$, where f is a real-valued function defined on the range of the vector of non-atomic measures $\mu = (\mu_1, \ldots, \mu_n)$ and continuous at $0 = \mu(\emptyset)$ and $\mu(I)$.*

(ii) *Moreover, there is a value of norm 1 on the closed space generated by $\mathcal{M}$ and Q.*

Our proof of Part (i) of Theorem 1 is self-contained (in section 3). It derives, however, from the ideas introduced in Mertens [6], which introduces the strictly stable distributions of index 1 into value theory.

Informally, the value constructed in the proof of Part (i) of Theorem 1 can be described by the following formula:

$$\varphi(f \circ \mu)(S) = \lim_{\delta \to 0+} \int \int_{3\delta}^{1-3\delta} f_{\mu(S)}(t\mu(I) + \delta^2 x) \, dt \, dP_\mu^\delta(x),$$

where P_μ^δ is the restriction of a strictly stable distribution of index 1, P_μ (on $\mathbb{R}^n$), to all points x for which there is a coalition S with $\delta x = 2\mu(S) - \mu(I)$. Thus our value averages the marginal contributions directly in a small neighborhood of the diagonal.

The Mertens value of a game $v \in Q \cap \mathcal{M}$ is obtained by first associating with v a game w that averages the marginal contributions at the diagonal and then averaging the derivatives of w in a neighborhood of the diagonal. The two approaches lead to the same value for market games (Proposition 4).

Section 2 reviews the basic concepts and results used in the proof of Part (i) of Theorem 1. Section 3 is the proof of Part (i) of Theorem 1. In Section 4 we prove that the value satisfies many other desirable properties in addition to the value axioms. Section 5 contains the proof of Part (ii) of Theorem 1, and comments on the fact that the countable additivity assumption on the measures appearing in the definition of the space Q can be replaced with finite additivity. Section 6 provides alternative formulas for the value in various special classes of games (Propositions 4 and 5), and additional approximations of the value (Proposition 3 and Lemma 10).

2. Preliminaries

In this section we review the basic concepts used in the proof of the main result.

2.1 NON-ATOMIC VECTOR MEASURES. Let $(I, \mathcal{C})$ be a measurable space. By a scalar measure (a measure for short) on $(I, \mathcal{C})$ we mean a countably additive function from $\mathcal{C}$ to $\mathbb{R}$. A finite dimensional vector measure (a vector measure for short) is a countably additive function from $\mathcal{C}$ to a finite dimensional real vector space, i.e., to $\mathbb{R}^n$. Any vector measure $\mu : \mathcal{C} \to \mathbb{R}^n$ is a vector of scalar measures $(\mu_1, \ldots, \mu_n)$. A vector measure $\mu = (\mu_1, \ldots, \mu_n)$ is non-atomic iff for every measurable S (i.e., $S \in \mathcal{C}$) with $\mu(S) \neq 0$ there is a measurable subset $T \subset S$ with $\mu(T) \neq 0$ and $\mu(T) \neq \mu(S)$. Equivalently, μ is non-atomic iff for every $1 \leq i \leq n$ the scalar measure μ_i is non-atomic.

The **range** of a vector measure μ, denoted $\mathcal{R}(\mu)$, is the set of all vectors $\mu(S)$ where $S \in \mathcal{C}$, i.e.,

$$\mathcal{R}(\mu) = \{\mu(S) : S \in \mathcal{C}\}.$$

The range of a non-atomic vector measure is convex and compact (Lyapunov's Theorem). The range of any vector measure is symmetric around $\mu(I)/2$, i.e., for every $x \in \mathcal{R}(\mu)$, there is a vector $y \in \mathcal{R}(\mu)$ $(y = \mu(I \setminus S))$ with $(x+y)/2 = \mu(I)/2$. Notice that it follows that if μ is a non-atomic vector measure, then $2\mathcal{R}(\mu) - \mu(I)$ is a convex subset of $\mathbb{R}^n$ that is centrally symmetric (around 0) and therefore its support function is a seminorm on $\mathbb{R}^n$. The convex hull of the range of a vector measure is the range of a non-atomic vector measure.

The seminorm $\| \ \|_\mu$ associated with the vector measure $\mu = (\mu_1, \ldots, \mu_n)$ is the support function of $2\mathcal{R}(\mu) - \mu(I)$, i.e., the function on $\mathbb{R}^n$ given by

$$\|y\|_\mu = \max\{\langle x, y\rangle \mid x \in 2\mathcal{R}(\mu) - \mu(I)\}$$

or equivalently

$$\|y\|_\mu = 2\max\{\langle \mu(S), y\rangle \mid S \in \mathcal{C}\} - \langle \mu(I), y\rangle.$$

Notice that:

(1) If $\mu_1, \ldots, \mu_n$ are independent, then $\| \ \|$ is a norm.

(2) If $\mu_1, \ldots, \mu_n$ are absolutely continuous w.r.t. the positive measure θ, then

$$\|y\|_\mu = \int |\sum_{i=1}^{n} (d\mu_i/d\theta) y_i| d\theta = \int |\langle d\mu/d\theta, y\rangle| d\theta.$$

(3) If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

$$\|y\|_{T\mu} = \|T^* y\|_\mu,$$

where T^* is the transpose of T, i.e., the map $T^*\colon \mathbb{R}^m \to \mathbb{R}^n$ such that for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, $\langle Tx, y\rangle = \langle x, T^* y\rangle$. Indeed,

$$\begin{aligned}\|y\|_{T\mu} &= 2\max\{\langle T\mu(S), y\rangle \mid S \in \mathcal{C}\} - \langle T\mu(I), y\rangle \\ &= 2\max\{\langle \mu(S), T^* y\rangle \mid S \in \mathcal{C}\} - \langle \mu(I), T^* y\rangle = \|T^* y\|_\mu.\end{aligned}$$

(4) If the vector measures $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$ have the same range, $\|y\|_\mu = \|y\|_\nu$, and thus in particular if $\theta \in \mathcal{G}$,

$$\|y\|_\mu = \|y\|_{\theta_* \mu},$$

where $\theta_* \mu = (\theta_* \mu_1, \ldots, \theta_* \mu_n)$.

2.2 The Cauchy distribution. The Cauchy distribution with parameter $\alpha > 0$ is the distribution on $\mathbb{R}$ with density $\alpha/\pi(\alpha^2 + x^2)$. If X and Y are independent Cauchy random variables with parameters α and β respectively and a and b are real numbers (with $a^2 + b^2 \neq 0$), then $aX + bY$ is a Cauchy random variable with parameter $|a|\alpha + |b|\beta$. It follows that if $X = (X_1, \ldots, X_k)$ is a vector of independent and identically distributed (i.i.d.) Cauchy random variables and $0 \neq y = (y_1, \ldots y_k) \in \mathbb{R}^k$, then the random variable $\langle y, X\rangle = \sum_{i=1}^k y_i X_i$ has the same distribution as the random variable $(\sum_{i=1}^k |y_i|)X_1$.

2.3 The characteristic function. The characteristic function of a probability distribution ν on $\mathbb{R}^n$ is the complex number-valued function φ: $\mathbb{R}^n \to \mathbb{C}$ given by

$$\varphi(y) = \int_{\mathbb{R}^n} \exp(i\langle y, x\rangle) d\nu(x).$$

The characteristic function of an $\mathbb{R}^n$-valued random variable X is the complex number valued function ψ: $\mathbb{R}^n \to \mathbb{C}$ given by

$$\psi(y) = E(i\langle y, X\rangle).$$

We recall here properties of the characteristic function.

(a) If φ is the characteristic function of the measure ν on a subspace V of $\mathbb{R}^n$ and $\varphi \in L_1(V)$, then ν is absolutely continuous w.r.t. the Lebesgue measure on V and its Radon—Nikodym derivative w.r.t. the Lebesgue measure is continuous.

(b) The characteristic function of the Cauchy distribution with parameter α is

$$\psi(t) = \exp(-\alpha|t|).$$

(c) If X_1 and X_2 are independent random variables with characteristic functions ψ_{X_1} and ψ_{X_2}, then the characteristic function of $X_1 + X_2$, $\psi_{X_1+X_2}$, is given by

$$\psi_{X_1+X_2}(y) = \psi_{X_1}(y)\psi_{X_2}(y).$$

(d) If X is an $\mathbb{R}^n$-valued random variable and T: $\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then TX is a $\mathbb{R}^m$-valued random variable with characteristic function

$$\psi_{TX}(y) = \psi_X(T^* y),$$

where T^* is the transpose of T. In particular, if $a \in \mathbb{R}$, $\psi_{aX}(y) = \psi_X(ay)$ and if X is a real-valued Cauchy random variable,

$$\psi_{aX}(y) = \psi_X(ay) = \exp(-|ay|);$$

and if $a \in \mathbb{R}^k$ and X is a vector of k i.i.d. Cauchy random variables with parameter α,

$$\psi_{\langle a,X\rangle}(y) = \exp\Big(- \alpha \sum_{i=1}^{k} |a_i y_i| \Big).$$

(e) If a sequence $(\psi_k)_{k=1}^{\infty}$ of characteristic functions converges pointwise to a function ψ that is continuous at 0, then ψ is a characteristic function.

2.4 Vector measures and characteristic functions. In this subsection we associate with every $\mathbb{R}^n$-valued vector measure μ a probability distribution P_μ on $\mathbb{R}^n$ and derive various relations. Given a vector measure μ, we denote by $AF(\mu)$ the affine space generated by $\mathcal{R}(\mu)$.

Lemma 1: *Let $\mu = (\mu_1, \dots, \mu_n)$ be a vector measure. Then the function $\varphi_\mu\colon \mathbb{R}^n \to \mathbb{R}$ given by*

$$\varphi_\mu(y) = \exp(-\|y\|_\mu)$$

is the characteristic function (Fourier transform) of a probability distribution P_μ on $AF(\mu)$. Moreover, P_μ is absolutely continuous w.r.t. the Lebesgue measure on $AF(\mu)$, and its Radon–Nikodym derivative w.r.t. the Lebesgue measure is continuous.

Proof: The function $\exp(-\|y\|_\mu)$ is continuous at 0 and integrable over $AF(\mu)$. Therefore it is sufficient to show that it is the pointwise limit of characteristic functions of probability measures on $AF(\mu)$. For any partition Π of $(I, \mathcal{C})$, let $(X_a)_{a\in\Pi}$ be a family of i.i.d. real-valued Cauchy distributions with characteristic function $\psi_{X_a}(t) = \exp(-|t|)$. Then the ($\mathbb{R}^n$-valued) random variable $\sum_{a\in\Pi} X_a \mu(a)$ takes values in $AF(\mu)$ and has a probability distribution ν_μ^Π whose characteristic function φ_μ^Π is given by

$$\varphi_\mu^\Pi(y) = \exp\Big(- \sum_{a\in\Pi} |\langle \mu(a), y\rangle| \Big).$$

A sequence of partitions $(\Pi_k)_{k=1}^{\infty}$ of $\mathcal{C}$ is called **admissible** if it is increasing and the σ-field generated by $\bigcup \Pi_k$ is $\mathcal{C}$. If $(\Pi_k)_{k=1}^{\infty}$ is an admissible sequence of partitions of $\mathcal{C}$, then

$$\lim_{k\to\infty} \sum_{a\in\Pi_k} |\langle \mu(a), y\rangle| = \|y\|_\mu$$

and thus

$$\lim_{k\to\infty} \varphi_\mu^{\Pi_k}(y) = \varphi_\mu(y),$$

and thus $\varphi_\mu(y)$ is indeed a characteristic function of a probability distribution P_μ on $\mathbb{R}^n$. ∎

An immediate corollary of the previous lemma asserts that a small translation in $AF(\mu)$ of P_μ is close in norm to P_μ. We denote the probability measure supported by a point $y \in \mathbb{R}^n$ by δ_y; and the convolution of two measures P and P' by $P * P'$. Notice that given a vector measure μ and a point $y \in AF(\mu)$, $(P_\mu * \delta_y)(A) = P_\mu(A - y)$ and thus the probability distribution $P_\mu * \delta_y$ is also absolutely continuous w.r.t. the Lebesgue measure on $AF(\mu)$ and its density at z is the density of the probability distribution P_μ at $z - y$. Thus, as P_μ has a continuous density,

COROLLARY 1: *For any vector measure $\mu = (\mu_1, \ldots, \mu_n)$ and $\varepsilon > 0$ there is $\delta > 0$ such that if $y \in AF(\mu)$ with $\|y\| < \delta$, then*

$$\|P_\mu - P_\mu * \delta_y\| < \varepsilon.$$

An additional useful relation is the following:

LEMMA 2: *If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then $P_{T\mu} = P_\mu \circ T^{-1}$.*

Proof: The characteristic function of $P_{T\mu}$ is

$$\varphi_{T\mu}(y) = \exp(-\|y\|_{T\mu}) = \exp(-\|T^*y\|_\mu)$$

and that of $P_\mu \circ T^{-1}$ is

$$\begin{aligned} E_{P_\mu \circ T^{-1}}(\exp(i\langle x, y\rangle)) &= E_{P_\mu}(\exp(i\langle Tx, y\rangle)) \\ &= E_{P_\mu}(\exp(i\langle x, T^*y\rangle) = \varphi_\mu(T^*y)) = \exp(-\|T^*y\|_\mu). \end{aligned}$$ ∎

The next lemma and corollary are not used in the proof of the main theorem. They are used in a later section discussing properties of the value constructed.

LEMMA 3: *Assume that $\mu = (\mu_1, \ldots, \mu_n)$ is an $\mathbb{R}^n$-valued vector measure and that $\nu^k = (\nu_1^k, \ldots, \nu_n^k)$ is a sequence of vector measures such that $AF(\nu^k) = AF(\mu)$ (from some k on), and $\|y\|_{\nu^k} \to_{k\to\infty} \|y\|_\mu$. Then*

$$\|P_\mu - P_{\nu^k}\| \to_{k\to\infty} 0.$$

Proof: As $AF(\nu^k) = AF(\mu)$ and $\|y\|_{\nu^k} \to_{k\to\infty} \|y\|_\mu$, we deduce that $\exp(-\|y\|_{\nu^k}) \to_{k\to\infty} \exp(-\|y\|_\mu)$ in $L_1(AF(\mu))$, and therefore

$$\|P_\mu - P_{\nu^k}\| \to_{k\to\infty} 0.$$ ∎

It follows in particular that

COROLLARY 2: *If μ is an $\mathbb{R}^n$-valued vector measure and Π_k is an admissible sequence of partitions and μ^k is the vector measure restricted to the field generated by Π_k, then $\|P_{\mu^k} - P_\mu\| \to 0$ as $k \to \infty$.*

3. Proof of Theorem 1

3.1 AN APPROXIMATE VALUE ON $Q(\mu)$. For any $\mathbb{R}^n$-valued non-atomic vector measure μ we define a map φ_μ^δ from $Q(\mu)$ — the space of all games of bounded variation that are functions of the vector measure μ and are continuous at $\mu(\emptyset)$ and at $\mu(I)$ — to BV. The map φ_μ^δ depends on a small positive constant $\delta > 0$ and the vector measure $\mu = (\mu_1, \ldots, \mu_n)$.

The linear space of games $Q(\mu)$ is not a symmetric space. Moreover, the map φ_μ^δ violates all value axioms. It does not map $Q(\mu)$ into FA, it is not efficient, and it is not symmetric. In addition, given two non-atomic vector measures, μ and ν, the operators φ_μ^δ and φ_ν^δ differ on the intersection $Q(\mu) \cap Q(\nu)$. However, it turns out that the violation of the value axioms by φ_μ^δ diminishes as δ goes to 0, and the difference $\varphi_\mu^\delta(f\circ\mu) - \varphi_\nu^\delta(g\circ\nu)$ goes to 0 as $\delta \to 0$ whenever $f\circ\mu = g\circ\nu$. Therefore, an appropriate limiting argument enables us to generate a value on the union of all the spaces $Q(\mu)$.

For $\delta > 0$ let $I_\delta(t) = I(3\delta \le t < 1 - 3\delta)$, where I stands for the indicator function. The essential role of the function I_δ is to make the integrands that appear in the integrals used in the definition of the value well-defined.

Let $\mu = (\mu_1, \ldots, \mu_n)$ be a vector of non-atomic probability measures and $f\colon \mathcal{R}(\mu) \to \mathbb{R}$ continuous at $\mu(\emptyset)$ and $\mu(I)$ and with $f \circ \mu$ of bounded variation.

It follows that for every $x \in 2\mathcal{R}(\mu) - \mu(I)$, $S \in \mathcal{C}$, and t with $I_\delta(t) = 1$, $t\mu(I) + \delta x$ and $t\mu(I) + \delta x + \delta\mu(S)$ are in $\mathcal{R}(\mu)$ and therefore (using Lyapunov's Theorem) the functions $t \mapsto I_\delta(t) f(t\mu(I) + \delta x)$ and $t \mapsto I_\delta(t) f(t\mu(I) + \delta x + \delta\mu(S))$ are of bounded variation on $[0, 1]$ and thus in particular they are integrable functions. Therefore, given a game $f \circ \mu \in Q(\mu)$, the function $F_{f,\mu}$, defined on all triples (δ, x, S) with $\delta > 0$ sufficiently small (e.g., $\delta < 1/9$), $x \in \mathbb{R}^n$ with $\delta x \in 2\mathcal{R}(\mu) - \mu(I)$, and $S \in \mathcal{C}$ by

$$F_{f,\mu}(\delta, x, S) = \int_0^1 I_\delta(t) \frac{f(t\mu(I) + \delta^2 x + \delta^3\mu(S)) - f(t\mu(I) + \delta^2 x)}{\delta^3} dt$$

is well-defined. Note that the continuity of f at $\mu(\emptyset)$ and $\mu(I)$ implies that

$$\text{(1)} \qquad \sup\{|F_{f,\mu}(\delta, x, I) - f(\mu(I))| \colon \delta x \in 2\mathcal{R}(\mu) - \mu(I)\} \to_{\delta \to 0+} 0.$$

That $F_{f,\mu}(\delta, x, S)$ is bounded on

$$\{(\delta, x, S) \colon \delta > 0, x \in \mathbb{R}^n, \delta x \in 2\mathcal{R}(\mu) - \mu(I), S \in \mathcal{C}\}$$

follows in particular from the next lemma that is used also in the sequel. For $\delta > 0$, $x \in 2\mathcal{R}(\mu) - \mu(I)$, and $S \in \mathcal{C}$, let

$$\begin{aligned} H(\delta, x, S) &= \frac{1}{\delta}\int_0^1 I_\delta(t)[f(t\mu(I) + \delta x + \delta\mu(S)) - f(t\mu(I) + \delta x)]dt \\ G(\delta, x, S) &= H(\delta, x + \mu(S), S^c) \\ &= \frac{1}{\delta}\int_0^1 I_\delta(t)[f(t\mu(I) + \delta x + \delta\mu(I)) - f(t\mu(I) + \delta x + \delta\mu(S))]dt. \end{aligned}$$

Lemma 4: *For sufficiently small $\delta > 0$, for every $x \in 2\mathcal{R}(\mu) - \mu(I)$ and $S \in \mathcal{C}$,*

$$\|f \circ \mu\| \geq |H(\delta, x, S)| + |G(\delta, x, S)|.$$

Proof: Let $K(\delta)$ be the smallest integer s.t. $3\delta + K(\delta)\delta > 1 - 3\delta$, and for $1 \leq k \leq K(\delta)$, $I_\delta^k \colon [0,1] \to \mathbb{R}$ is defined by

$$I_\delta^k(t) = I(3\delta + (k-1)\delta \leq t < 3\delta + k\delta).$$

Then $I_\delta(t) \leq \sum_{k=1}^{K(\delta)} I_\delta^k(t)$. Let $T_x \in \mathcal{C}$ with $x = 2\mu(T_x) - \mu(I)$. In what follows we identify a coalition $S \in \mathcal{C}$ with its characteristic function. For every $0 \leq \alpha < \delta$, consider the increasing chain of ideal coalitions, h_k^α, $0 \leq k \leq 2K(\delta)$, where $h_0^\alpha = 3\delta + \alpha + \delta(2T_x - 1)$, $h_{2k+1}^\alpha = h_{2k}^\alpha + \delta S$, and $h_{2k}^\alpha = h_0^\alpha + k\delta$. Then

$$\begin{aligned} |H(\delta, x, S)| &\leq \frac{1}{\delta}\int_0^1 \sum_{k=1}^{K(\delta)} I_\delta^k(t)|f(t\mu(I) + \delta x + \delta\mu(S)) - f(t\mu(I) + \delta x)|dt \\ &= \frac{1}{\delta}\int_0^\delta \sum_{k=1}^{K(\delta)} |f(\mu(h_{2k-1}^\alpha)) - f(\mu(h_{2(k-1)}^\alpha))|d\alpha, \\ |G(\delta, x, S)| &\leq \frac{1}{\delta}\int_0^\delta \sum_{k=1}^{K(\delta)} |f(\mu(h_{2k}^\alpha)) - f(\mu(h_{2k-1}^\alpha))|d\alpha. \end{aligned}$$

Therefore,

$$|H(\delta, x, S)| + |G(\delta, x, S)| \leq \frac{1}{\delta}\int_0^\delta \sum_{k=1}^{2K(\delta)} |f(\mu(h_k^\alpha)) - f(\mu(h_{k-1}^\alpha))|d\alpha.$$

As for every $0 \leq \alpha < \delta$, $\sum_{k=1}^{2K(\delta)} |f(\mu(h_k^\alpha)) - f(\mu(h_{k-1}^\alpha))| \leq \|f \circ \mu\|_{BV}$, it follows that $\|f \circ \mu\|_{BV} \geq |H(\delta, x, S)| + |G(\delta, x, S)|$, which completes the proof of the lemma. ∎

That $F_{f,\mu}(\delta, x, S)$ is continuous in x follows from the following lemma.

LEMMA 5: *For sufficiently small $\delta > 0$, $H(\delta, x, S)$ is continuous in x.*

Proof: Notice that for $x \in 2\mathcal{R}(\mu) - \mu(I)$ and $S \in \mathcal{C}$, $x \in 3\mathcal{R}(\mu) - \mu(I)$ and $x + \mu(S) \in 3\mathcal{R}(\mu) - \mu(I)$. Therefore it is enough to prove that for sufficiently small $\delta > 0$,

$$\int_0^1 I_\delta(t) f(t\mu(I) + \delta x) dt$$

is continuous in x on $3\mathcal{R}(\mu) - \mu(I)$. Fix $x \in 3\mathcal{R}(\mu) - \mu(I)$. It is sufficient to prove that for a.e. t in $[0,1]$

(2) $$I_\delta(t) f(t\mu(I) + \delta x) \text{ is continuous in } x \in 3\mathcal{R}(\mu) - \mu(I).$$

We will show that (2) holds for all but countably many values of t in $[0,1]$. Otherwise, there is $\theta > 0$ s.t. for every m there is an increasing sequence

$$3\delta < t_1 < t_2 < \cdots t_m < 3\delta + K(\delta)\delta$$

and sequences $x_{k,i}$, $1 \le i \le m$, $k \ge 1$, with $x_{k,i} \to_{k\to\infty} x$ with

$$\limsup_{k\to\infty} |f(t_i\mu(I) + \delta x_{k,i}) - f(t_i\mu(I) + \delta x)| > \theta > 0$$

for every $1 \le i \le m$; and then w.l.o.g. we assume (by possibly taking a subsequence) that the following limits and equalities exist:

$$\lim_{k\to\infty} |f(t_i\mu(I) + \delta x_{k,i}) - f(t_i\mu(I) + \delta x)| = \theta_i > \theta > 0.$$

Fix $g \in \mathcal{B}(I, \mathcal{C})$ with $\mu(g) = x$ and $\|g\| \le 3$. Then, for every $1 \le i \le m$, there exists a sequence $g_{k,i} \in \mathcal{B}(I, \mathcal{C})$ with $\|g_{k,i} - g\| \to_{k\to\infty} 0$ and $\mu(g_{k,i}) = x_{k,i}$. Consider the following increasing chain:

$$0 \le h_i = t_i + \delta\left[(g_{k,i} \wedge g) + (1+\epsilon_i)(g_{k,i} - g)^+/2 + (1-\epsilon_i)(g - g_{k,i})^+/2\right] \le 1,$$

where $\epsilon_1, \ldots, \epsilon_k$ is a given sequence of signs ± 1. Note that for $\epsilon_i = 1$, $f(\mu(h_i)) = f(t_i\mu(I) + \delta x_{k,i})$; and for $\epsilon_i = -1$, $f(\mu(h_i)) = f(t_i\mu(I) + \delta x)$. Therefore, given $1 \le i \le m$, $k \ge 1$ and $0 \le h \le 1$, we can choose ϵ_i so that

$$|f(\mu(h_i)) - f(\mu(h))| \ge |f(t_i\mu(I) + \delta x_{k,i}) - f(t_i\mu(I) + \delta x)|/2.$$

Therefore, for a sufficiently large k, for an appropriate choice of signs ϵ_i the chain $0 \le h_1 \le \cdots \le h_m \le 1$ satisfies

$$\sum |f(\mu(h_i)) - f(\mu(h_{i-1}))| \ge m\theta/2.$$

By [2, p. 66, Theorem 4] there is an increasing chain of coalitions T_i, $1 \le i \le k$, with $\mu(h_i) = \mu(T_i)$ and thus

$$\sum |f(\mu(T_i)) - f(\mu(T_{i-1}))| \ge m\theta/2,$$

which contradicts the bounded variation of $f \circ \mu$. ∎

Let P_μ^δ be the restriction of P_μ to the set of all points in

$$\{x \in \mathbb{R}^n : \delta x \in 2\mathcal{R}(\mu) - \mu(I)\}.$$

The continuity and boundedness of $F_{f,\mu}(\delta, x, S)$ in x implies that the function φ_μ^δ defined on $Q(\mu) \times \mathcal{C}$ by

$$\varphi_\mu^\delta(f \circ \mu, S) = \int_{AF(\mu)} F_{f,\mu}(\delta, x, S) dP_\mu^\delta(x)$$

is well-defined.

The next lemma states that the quantified violation of the value axioms by φ_μ^δ goes to zero as δ goes to zero.

LEMMA 6: *For every $v = f \circ \mu, u = g \circ \mu \in Q(\mu)$ and $S, T \in \mathcal{C}$ with $S \cap T = \emptyset$,*

$$\varphi_\mu^\delta(v, S) + \varphi_\mu^\delta(u, S) = \varphi_\mu^\delta(v + u, S), \tag{3}$$

$$\varphi_\mu^\delta(v, \theta_* S) = \varphi_{\theta_*\mu}^\delta(\theta_* v, S), \tag{4}$$

$$\varphi_\mu^\delta(v, I) \to_{\delta\to 0} v(I), \tag{5}$$

$$\varphi_\mu^\delta(v, S) + \varphi_\mu^\delta(v, T) - \varphi_\mu^\delta(v, S \cup T) \to_{\delta \to 0} 0, \tag{6}$$

$$\limsup_{\delta \to 0} |\varphi_\mu^\delta(v, S)| + |\varphi_\mu^\delta(v, S^c)| \le \|v\|. \tag{7}$$

Proof: Equality (3) follows from the equality $F_{f,\mu} + F_{g,\mu} = F_{f+g,\mu}$ and the definition of φ_μ^δ as an integral. Note that $\mathcal{R}(\mu) = \mathcal{R}(\theta_*\mu)$ and thus $P_\mu^\delta = P_{\theta_*\mu}^\delta$. Thus, Equality (4) follows from the equality $F_{f,\mu}(\delta, x, \theta S) = F_{f,\theta_*\mu}(\delta, x, S)$ and the definition of φ_μ^δ as an integral. The approximate efficiency, (5), follows from (1) and the definitions of φ_μ^δ. The limiting results (6) and (7) use Corollary 1. Indeed, note that $\|P_\mu^\delta * \delta_{-\delta\mu(S)} - P_\mu * \delta_{-\delta\mu(S)}\| = \|P_\mu^\delta - P_\mu\|$, and therefore by the triangle inequality

$$\|P_\mu^\delta * \delta_{-\delta\mu(S)} - P_\mu^\delta\| \le 2\|P_\mu^\delta - P_\mu\| + \|P_\mu * \delta_{-\delta\mu(S)} - P_\mu\|.$$

Applying Corollary 1 we deduce that

$$\|P_\mu^\delta * \delta_{-\delta\mu(S)} - P_\mu^\delta\| \to_{\delta \to 0+} 0. \tag{8}$$

On the other hand,

$$F_{f,\mu}(\delta, x, S \cup T) = F_{f,\mu}(\delta, x, S) + F_{f,\mu}(\delta, x + \delta\mu(S), T)$$

and therefore, if $v = f \circ \mu$,

$$\begin{aligned}\varphi_\mu^\delta(v, S \cup T) = & \int_{AF(\mu)} F_{f,\mu}(\delta, x, S \cup T) dP_\mu^\delta(x) \\ = & \varphi_\mu^\delta(v, S) + \int_{AF(\mu)} F_{f,\mu}(\delta, x + \delta\mu(S), T) dP_\mu^\delta(x) \\ = & \varphi_\mu^\delta(v, S) + \int_{AF(\mu)} F_{f,\mu}(\delta, x, T) d(P_\mu^\delta * \delta_{-\delta\mu(S)})(x) \\ & + \varphi_\mu^\delta(v, T) - \int_{AF(\mu)} F_{f,\mu}(\delta, x, T) dP_\mu^\delta(x),\end{aligned}$$

which together with (8) implies (6). Similarly, as

$$|F_{f,\mu}(\delta, x, S)| + |F_{f,\mu}(\delta, x + \delta\mu(S), S^c)| \le \|v\|$$

by Lemma 4,

$$\begin{aligned}\|v\| \ge & \int | \int_{AF(\mu)} F_{f,\mu}(\delta, x, S) dP_\mu^\delta(x)| + | \int_{AF(\mu)} F_{f,\mu}(\delta, x + \delta\mu(S), S^c) dP_\mu^\delta(x)| \\ = & \int |\varphi_\mu^\delta(v, S)| + | \int_{AF(\mu)} F_{f,\mu}(\delta, x, S^c) d(P_\mu^\delta * \delta_{-\delta\mu(S)})(x)| \\ & + |\varphi_\mu^\delta(v, S^c)| - | \int_{AF(\mu)} F_{f,\mu}(\delta, x, S^c) dP_\mu^\delta(x)|,\end{aligned}$$

which together with (8) implies (7). ∎

The next lemma illustrates that the dependence of the representation diminishes as δ goes to 0.

LEMMA 7: *If $v \in Q(\mu)$ and $v \in Q(\nu)$ and $S \in \mathcal{C}$, then*

$$|\varphi_\mu^\delta(v, S) - \varphi_\nu^\delta(v, S)| \to_{\delta \to 0} 0.$$

Proof: It is sufficient to prove the lemma in the case that $\nu = (\nu_1, \ldots, \nu_n)$, $\mu = (\nu_1, \ldots, \nu_n, \mu_{n+1}, \ldots, \mu_k)$ and $v = f \circ \mu = g \circ \nu$. Consider the projection T from $\mathbb{R}^k$ onto the first n coordinates. It follows that $T\mu = \nu$. Note that for every $x \in \mathbb{R}^k$ with $\delta x \in 2\mathcal{R}(\mu) - \mu(I)$, $F_{f,\mu}(\delta, x, S) = F_{g,\nu}(\delta, Tx, S)$. Using the definition of φ_μ^δ,

$$\begin{aligned}\varphi_\mu^\delta(f\circ\mu)(S) &= \int_{AF(\mu)} F_{f,\mu}(\delta,x,S)dP_\mu^\delta(x)\\ &= \int_{AF(\mu)} F_{g,\nu}(\delta,Tx,S)dP_\mu^\delta(x)\\ &= \int_{AF(\nu)} F_{g,\nu}(\delta,x,S)d(P_\mu^\delta\circ T^{-1})(x).\end{aligned}$$

On the other hand, using the definition of φ_ν^δ,

$$\varphi_\nu^\delta(g\circ\nu)(S) = \int_{AF(\nu)} F_{g,\nu}(\delta,x,S)dP_\nu^\delta(x).$$

It is thus sufficient to prove that

$$\|P_\mu^\delta\circ T^{-1} - P_\nu^\delta\| \to_{\delta\to 0} 0.$$

As $\{x\in\mathbb{R}^n : \delta x\in 2\mathcal{R}(\mu)-\mu(I)\}$ increases to $AF(\mu)$ as $\delta\to 0+$, $\|P_\mu^\delta - P_\mu\|\to 0$ as $\delta\to 0$. By Lemma 3, $P_{T\mu} = P_\mu\circ T^{-1}$ for every linear map T: $\mathbb{R}^k\to\mathbb{R}^n$. As $\|P_\mu^\delta\circ T^{-1} - P_\mu\circ T^{-1}\|\le\|P_\mu^\delta - P_\mu\|$, we have

$$\|P_\mu^\delta\circ T^{-1} - P_{T\mu}^\delta\| \le \|P_\mu^\delta - P_\mu\| + \|P_{T\mu}^\delta - P_{T\mu}\| \to_{\delta\to 0+} 0.$$

Therefore,

$$\|P_\mu^\delta\circ T^{-1} - P_{T\mu}^\delta\| \to_{\delta\to 0} 0. \qquad \blacksquare$$

3.2 A VALUE OF NORM 1 ON $\bar{Q}$. Lemma 7 enables us to define a map φ: $Q\to\mathbb{R}^{\mathcal{C}}$ as a "limit" of the maps φ_μ^δ as $\delta\to 0$. Consider the partially ordered linear space $\mathcal{L}$ of all bounded functions defined on the open interval $(0,1/9)$ with the partial order $h\succ g$ iff $h(\delta)\ge g(\delta)$ for all sufficiently small values of $\delta>0$. Let L: $\mathcal{L}\to\mathbb{R}$ be a monotonic (i.e., $L(h)\ge L(g)$ whenever $h\succ g$) linear functional with $L(\mathbf{1})=1$. It follows in particular that for every $h\in\mathcal{L}$,

$$\liminf_{\delta\to 0+} h(\delta)\le L(h)\le\limsup_{\delta\to 0+} h(\delta).$$

Define the map φ: $Q\to\mathbb{R}^{\mathcal{C}}$ by

$$\varphi v(S) = L(\varphi_\mu^\delta(v,S))$$

whenever $v\in Q(\mu)$. That φ is well-defined follows from Lemma 7 and Part 7 of Lemma 6. That φv is in FA and that φ is a value of norm 1 on Q follows from Lemma 6. The continuous extension of φ to $\bar{Q}$ is also denoted by φ. As φ is a value of norm 1 on Q and the continuous extension of any value of norm 1 defines a value (of norm 1) on the closure, we have:

PROPOSITION 1: *φ is a value of norm 1 on $\bar{Q}$.*

4. Additional properties of φ

4.1 THE RANGE OF φ. It is known [4] that a value of a non-atomic vector measure game need not be a linear combination of the measures defining the game. We will show in the next lemma that the value φv of a game $v \in Q$ is a linear combination of the measures defining the game and thus, in particular, $\varphi v \in NA$. As $\varphi\colon \bar{Q} \to FA$ is continuous and NA is closed in BV, $\varphi v \in NA$ for every game $v \in \bar{Q}$.

LEMMA 8: *Let $\mu = (\mu_1, \ldots, \mu_n)$ be a vector of non-atomic probability measures, and $f : \mathcal{R}(\mu) \to \mathbb{R}$ with $f \circ \mu \in Q(\mu)$. Then $\varphi(f \circ \mu)$ is a linear combination of the measures μ_i, $i = 1, \ldots, n$, i.e., there are numbers $a_i(f, \mu)$ such that*

$$\varphi(f \circ \mu) = \sum_{i=1}^{n} a_i(f, \mu)\mu_i.$$

Proof: It follows from the definition of φ that for any two coalitions $S, T \in \mathcal{C}$ with $\mu(S) = \mu(T)$,

$$\varphi(f \circ \mu)(S) = \varphi(f \circ \mu)(T).$$

Therefore, the value φ induces a map $T\colon \mathcal{R}(\mu) \to \mathbb{R}$; if $x = \mu(S) \in \mathcal{R}(\mu)$, then $T(x)$ is given by $T(x) = \varphi(f \circ \mu)(S)$. As $\varphi(f \circ \mu) \in FA$, it is finitely additive and bounded. Thus

$$T\Big(\frac{x+y}{2}\Big) = \frac{T(x) + T(y)}{2}$$

by the finite additivity which, together with the boundedness, implies that T is linear on $\mathcal{R}(\mu)$ and therefore there are constants a_i, $i = 1, \ldots, n$, such that $T(x) = \sum_{i=1}^{n} a_i x_i$, implying that $\varphi(f \circ \mu) = \sum_{i=1}^{n} a_i \mu_i$. ∎

4.2 DUALITY. The dual of a game v is the game v^* defined by $v^*(S) = v(I) - v(I \setminus S)$. If V is a symmetric linear space of games, so are $V^* = \{v^* : v \in V\}$ and $U = \{v + v^* : v \in V\}$. A set of games is called self-dual if $V = V^*$. A map φ defined on a self-dual set of games V is self-dual if $\varphi v = \varphi v^*$ for every $v \in V$. If φ is a value on V, then the map $\varphi^*\colon V^* \to FA$, called the dual of φ and defined by $\varphi^* v^* = \varphi v$, is a value on V^*. If ψ is a value on U, the map φ: $V \to FA$, defined by

$$\varphi v = \frac{\psi v + \psi v^*}{2},$$

is a self-dual value on V.

The space Q is self-dual, and thus also its closure is self-dual. Indeed, if f is defined on the range of a vector of non-atomic probability measures μ, the function f^* defined by $f^*(x) = f(\mu(I)) - f(\mu(I) - x)$ is well-defined on the range of μ ($\mu(I) - x \in \mathcal{R}(\mu)$ whenever $x \in \mathcal{R}(\mu)$). In addition, if f is continuous at $\mu(\emptyset)$ and at $\mu(I)$, so is f^*, and if $f \circ \mu$ is of bounded variation, so is $f^* \circ \mu$ ($\|f^* \circ \mu\| = \|f \circ \mu\|$). As $(f \circ \mu)^* = f^* \circ \mu$, the space Q is self-dual. Finally, as the map $v \to v^*$ is an isometry, the closure of a self-dual set of games is self-dual.

LEMMA 9: *The value φ on $\bar{Q}$ is self-dual.*

Proof: Assume that $v = f \circ \mu \in Q(\mu)$. Note that for sufficiently small values of $\delta > 0$, $F_{f,\mu}(\delta, x, S) = F_{f^*,\mu}(\delta, -x - \delta\mu(S), S)$ whenever $\delta x \in 2\mathcal{R}(\mu) - \mu(I)$ and $S \in \mathcal{C}$. As P_μ^δ is symmetric ($P_\mu^\delta(A) = P_\mu^\delta(-A)$), and $\|P_\mu^\delta - P_\mu^\delta * \delta_{\delta\mu(S)}\| \to 0$ as $\delta \to 0$, we deduce that $\varphi_\mu^\delta(f \circ \mu, S) - \varphi_\mu^\delta(f^* \circ \mu, S)$ goes to 0 as $\delta \to 0$ and therefore $\varphi v = \varphi v^*$ for every $v \in Q$. As φ is continuous and the duality map $v \to v^*$ is an isometry, $\varphi v = \varphi v^*$ for every game v in the closure of Q. ∎

4.3 STRONG POSITIVITY. A desirable property of a value is strong positivity [8]. Given two games v and u and a coalition S, we write $v \succ_S u$ iff $v(T \cup S') \geq u(T \cup S')$ for every $S' \subset S$ and T in $\mathcal{C}$. Given a set of games V, a map φ: $V \to \mathbb{R}^{\mathcal{C}}$ is **strongly positive** if $\varphi v(S) \geq \varphi u(S)$ whenever $S \in \mathcal{C}$, $v, u \in V$ and $v \succ_S u$.

We demonstrate now that φ: $Q \to FA$ is strongly positive. Assume that $v, u \in Q$ and $S \in \mathcal{C}$ with $v \succ_S u$. There exist a vector of non-atomic probability measures μ and real-valued functions f and g defined on the range of μ such that $v = f \circ \mu$ and $u = g \circ \mu$. As $v \succ_S u$, we have

$$f(t\mu(I)+\delta^2 x+\delta^3\mu(S))-f(t\mu(I)+\delta^2 x)) \geq g(t\mu(I)+\delta^2 x+\delta^3\mu(S))-g(t\mu(I)+\delta^2 x))$$

whenever $0 < \delta < t < 1 - 3\delta$ and $x \in AF(\mu)$ with $\delta x \in 2\mathcal{R}(\mu) - \mu(I)$. Therefore, $F_{f,\mu}(\delta, x, S) \geq F_{g,\mu}(\delta, x, S)$ and thus $\varphi_\mu^\delta(f \circ \mu, S) \geq \varphi_\mu^\delta(g \circ \mu, S)$, implying that $\varphi(f \circ \mu)(S) \geq \varphi(g \circ \mu)(S)$. Therefore, φ: $Q \to FA$ is strongly positive.

We prove in this subsection that the extension of φ to $\bar{Q}$ is strongly positive. Note that a strongly positive value (of norm 1) on a space Q need not have an extension to a strongly positive value on the closure of Q. We thus take an alternative route, proving that φ obeys a new and stronger property than strong positivity on Q. The stronger property enables us to prove that the extension of φ to $\bar{Q}$ is strongly positive.

PROPOSITION 2: *φ is strongly positive on the closure of Q.*

Proof: Assume that $v, u \in \bar{Q}$ and $S \in \mathcal{C}$ with $v \succ_S u$. Fix $\varepsilon > 0$. There exist a vector of non-atomic probability measures, μ, and real-valued functions f and g defined on $\mathcal{R}(\mu)$ with $f \circ \mu, g \circ \mu \in Q(\mu)$ and s.t. $\|v - f \circ \mu\| < \varepsilon$ and $\|u - g \circ \mu\| < \varepsilon$.

We show first that for $\delta > 0$ sufficiently small and $x \in AF(\mu)$ with $\delta x \in 2\mathcal{R}(\mu) - \mu(I)$,

$$F_{f,\mu}(\delta, x, S) - F_{g,\mu}(\delta, x, S) \geq -4\varepsilon.$$

Let $T_{\delta,x} \in \mathcal{C}$ with $\delta x = 2\mu(T_{\delta,x}) - \mu(I)$. Let $K(\delta)$ be the largest positive integer so that $I_\delta(3\delta + K(\delta)\delta^3) = 1$. For every $0 \leq \alpha < \delta^3$, consider the increasing chain of ideal coalitions, h_k^α, $0 \leq k \leq 2K(\delta)$, where $h_0^\alpha = 3\delta + \alpha + \delta(2T_{\delta,x} - 1)$, $h_{2k+1}^\alpha = h_{2k}^\alpha + \delta^3 S$, and $h_{2k}^\alpha = h_0^\alpha + k\delta^3$. Using the Dvoretzky–Wald–Wolfowitz Theorem [2], there is an increasing sequence of coalitions T_k^α, $0 \leq k \leq 2K(\delta)$, such that $\mu(T_k^\alpha) = \mu(h_k^\alpha)$ and $T_{2k+1}^\alpha \setminus T_{2k}^\alpha \subset S$. Therefore, as $\|v - f \circ \mu\| < \varepsilon$ and $\|u - g \circ \mu\| < \varepsilon$,

$$\begin{aligned}
\sum_{0 \leq k < K(\delta)} \left(f(\mu(h_{2k+1}^\alpha)) - f(\mu(h_{2k}^\alpha))\right) &= \sum_{0 \leq k < K(\delta)} \left(f(\mu(T_{2k+1}^\alpha)) - f(\mu(T_{2k}^\alpha))\right) \\
&\geq \sum_{0 \leq k < K(\delta)} \left(v(T_{2k+1}^\alpha) - v(T_{2k}^\alpha)\right) - \varepsilon \\
&\geq \sum_{0 \leq k < K(\delta)} \left(u(T_{2k+1}^\alpha) - u(T_{2k}^\alpha)\right) - \varepsilon \\
&\geq \sum_{0 \leq k < K(\delta)} \left(g(\mu(T_{2k+1}^\alpha)) - g(\mu(T_{2k}^\alpha))\right) - 2\varepsilon \\
&= \sum_{0 \leq k < K(\delta)} \left(g(\mu(h_{2k+1}^\alpha)) - g(\mu(h_{2k}^\alpha))\right) - 2\varepsilon.
\end{aligned}$$

Using the continuity of f and g at $\mu(I)$, we deduce that for sufficiently small values of $\delta > 0$, for every $3\delta + (K(\delta) - 1)\delta^3 \leq t$ with $I_\delta(t) = 1$,

$$|f(t\mu(I) + \delta^2 x + \delta^3 \mu(S)) - f(t\mu(I) + \delta^2 x)| < \varepsilon$$

and

$$|g(t\mu(I) + \delta^2 x + \delta^3 \mu(S)) - g(t\mu(I) + \delta^2 x)| < \varepsilon.$$

Therefore,

$$F_{f,\mu}(\delta, x, S) \geq \int_0^{\delta^3} \sum_{0 \leq k < K(\delta)} \frac{f(\mu(h_{2k+1}^\alpha)) - f(\mu(h_{2k}^\alpha))}{\delta^3} d\alpha - \varepsilon$$

and

$$F_{g,\mu}(\delta, x, S) \leq \int_0^{\delta^3} \sum_{0 \leq k < K(\delta)} \frac{g(\mu(h^\alpha_{2k+1})) - g(\mu(h^\alpha_{2k}))}{\delta^3} d\alpha + \varepsilon.$$

We conclude that for sufficiently small values of $\delta > 0$,

$$F_{f,\mu}(\delta, x, S) \geq F_{g,\mu}(\delta, x, S) - 4\varepsilon$$

for every $x \in AF(\mu)$ with $\delta x \in 2\mathcal{R}(\mu) - \mu(I)$. Therefore $\varphi^\delta_\mu(f \circ \mu, S) \geq \varphi^\delta_\mu(g \circ \mu, S) - 4\varepsilon$ and thus $\varphi(f \circ \mu)(S) \geq \varphi(g \circ \mu)(S) - 4\varepsilon$. Together with the inequalities $\|\varphi\| \leq 1$, $\|v - f \circ \mu\| < \varepsilon$, and $\|u - g \circ \mu\| < \varepsilon$, it implies that $\varphi v(S) \geq \varphi u(S) - 6\varepsilon$. As this holds for every $\varepsilon > 0$, we conclude that $\varphi v(S) \geq \varphi u(S)$. ∎

4.4 DIAGONALITY. There are non-diagonal values [11]. However, a continuous value is diagonal [9] and thus φ is a diagonal value of norm 1 on $\bar{Q}$. Moreover, it is clear from our proof that our value has an extension to include also $DIAG$ in its domain.

4.5 UNIQUENESS. Our value on Q depends on the functional L used in the definition of φ. The non-uniqueness of such a positive linear functional L on $\mathcal{L}$ illustrates in particular that there are many values of norm 1 on Q.

Consider the subspace $Q_1 \subset Q$ that consists of all games $v = f \circ \mu \in Q$ for which the limit of $\varphi^\delta_\mu(v, S)$ as $\delta \to 0$ exists. The restriction of our value to Q_1 is thus independent of the choice of the linear functional L. We do not know if there is more than one strongly positive value of norm 1 on $\bar{Q}_1$. If there is more than one, it will be interesting to specify a large subspace of Q_1 that has a unique strongly positive value of norm 1.

5. Extensions and variations

5.1 VALUES OF FA VECTOR MEASURES GAMES. We are dropping now the countable additivity assumption on the vector measures appearing in the definition of the space Q. We recall first the definition of a non-atomic finitely additive measure. A finitely additive measure $\mu \in FA^+$ is **non-atomic** if for every $S \in \mathcal{C}$ with $\mu(S) > 0$ there is a subset of S, $T \in \mathcal{C}$, with $\mu(S)/3 < \mu(T) < 2\mu(S)/3$. The range of a vector of finitely additive non-atomic measures is convex and the Dvoretzky–Wald–Wolfowitz Theorem [2] holds for a vector of finitely additive non-atomic measures $\mu = (\mu_1, \ldots, \mu_n)$: for every increasing sequence of ideal coalitions $0 \leq h_1 \leq \cdots \leq h_k \leq \mathbf{1}$ there exists an increasing sequence of coalitions

$\emptyset \subset S_1 \subset \cdots \subset S_k \subset I$ such that $\mu(h_i) = \mu(S_i)$. Therefore, our proof holds also for the space of all vector measure games of bounded variation $f \circ \mu$, where μ is a vector of non-atomic finitely additive measures and f is defined on the range of μ and continuous at $\mu(\emptyset)$ and $\mu(I)$.

THEOREM 2: *There is a (strongly positive diagonal and self-dual) value of norm 1 on the closed subspace of BV that is generated by all games of the form $f \circ \mu$, where μ is a vector of finitely additive non-atomic measures $\mu = (\mu_1, \ldots, \mu_n)$ and f is a real-valued function defined on the range of μ, continuous at $\mu(\emptyset)$ and $\mu(I)$.*

5.2 PROOF OF PART (ii) THEOREM 1. Mertens [6] constructs a value ψ on a large space $\mathcal{M}$ that includes all games of the form $f \circ \mu$ where $f \in bv'$ and $\mu \in FA^1$, the algebra generated by $bv'NA$, and all games generated by lattice and algebraic operations from a finite number of measures. It is of interest to investigate the relation between the Mertens value ψ and our value φ on $\mathcal{M} \cap \bar{Q}$. We do not have an example of a game $v \in \mathcal{M} \cap \bar{Q}$ where $\psi v \neq \varphi v$.

One can show, on the one hand, that our value φ on Q has an extension to a value on a space that includes $\mathcal{M}$. On the other hand, we will show that the Mertens value ψ has an extension to a value of norm 1 on a larger space that includes Q.

The Mertens value is defined by means of three maps φ_1, φ_2, and φ_3. The first operator, φ_1, maps every game v to the constant sum game $\frac{1}{2}(v + v^*)$. The map φ_2, called the Mertens extension operator, maps a game v in its domain, $\mathrm{Dom}(\varphi_2)$, to a function $\bar{v}$ defined on $B_1^+(I, \mathcal{C})$. Given $\tau > 0$ sufficiently small, define

$$[\varphi_3^\tau(v)](\chi) = \int_0^1 \frac{\bar{v}(t + \tau\chi) - \bar{v}(t - \tau\chi)}{2\tau}\, dt$$

where, for $f \in B(I, \mathcal{C})$, $\bar{v}(f) = \bar{v}[\max(0, \min(\mathbf{1}, f))]$. The map φ_3, called the Mertens value derivative, is defined as

$$\begin{aligned}[\varphi_3(v)](\chi) &= \lim_{\tau \to 0+} [\varphi_3^\tau(v)](\chi) \\ &= \lim_{\tau \to 0+} \int_0^1 \frac{\bar{v}(t + \tau\chi) - \bar{v}(t - \tau\chi)}{2\tau} dt\end{aligned}$$

on its domain $\mathrm{Dom}(\varphi_3)$. For simplicity, we only consider games of bounded variation with $\lim_{\tau \to 0+} \bar{v}(\tau\chi) = 0\ \forall \chi \in B_1^+(I, \mathcal{C})$ (and thus if v is constant sum $\lim_{\tau \to 0+} \bar{v}(\mathbf{1} - \tau\chi) = \bar{v}(\mathbf{1})\ \forall \chi \in B_1^+(I, \mathcal{C})$). Under these assumptions the integrals $[\varphi_3^\tau(v)](\chi)$ exist for all $\chi \in B(I, \mathcal{C})$. The domain of φ_3 consists of all constant sum games for which the limit of $[\varphi_3^\tau(v)](\chi)$ as $\tau \to 0+$ exists for every $\chi \in B_1^+(I, \mathcal{C})$.

Note that for a game $v \in Q$ the integral

$$\int_0^1 \frac{\bar{v}(t+\tau\chi) - \bar{v}(t-\tau\chi)}{2\tau} dt$$

exists for every $\chi \in B_1^+(I,\mathcal{C})$, but the limit of $[\varphi_3^\tau(v)](\chi)$ as $\tau \to 0+$ need not exist. We modify the definition of φ_3 so that the modified map will be defined even if the limit of $[\varphi_3^\tau(v)](\chi)$ as $\tau \to 0+$ does not exist. Let $\mathcal{L}$ be the ordered linear space of germs of bounded functions defined in a right neighborhood of 0. A germ is an equivalence class of functions defined in a right neighborhood of 0; two functions are equivalent if they agree in some right neighborhood of 0. Given $h, g \in \mathcal{L}$, $h \succ g$ iff there is $\theta > 0$ such that h and g are defined on $(0,\theta)$ and for every $0 < x < \theta$, $h(x) \geq g(x)$. The semi-group of positive numbers with multiplication acts on $\mathcal{L}$ as follows: Given $a > 0$ and $h \in \mathcal{L}$ (defined on $(0,\theta)$), $a * h \in \mathcal{L}$ (defined on $(0,\theta/a)$) is given by $(a*h)(x) = h(ax)$. A linear functional $L\colon \mathcal{L} \to \mathbb{R}$ is **scale invariant** if for every $a > 0$ and $h \in \mathcal{L}$, $L(a*h) = L(h)$. Fix a positive (i.e., $L(h) \geq L(g)$ whenever $h \succ g$) scale invariant linear functional $L\colon \mathcal{L} \to \mathbb{R}$ with $L(\mathbf{1}) = 1$.

Define the map φ_3^L by

$$\begin{aligned}[\varphi_3^L(v)](\chi) =& L([\varphi_3^\tau(v)](\chi)) \\ =& L\left(\int_0^1 \frac{\bar{v}(t+\tau\chi) - \bar{v}(t-\tau\chi)}{2\tau} dt\right).\end{aligned}$$

It is obvious from the scale invariance of L that for every b,

$$[\varphi_3^L(v)](b\chi) = b[\varphi_3^L(v)](\chi). \tag{9}$$

As in Mertens [6], one shows that

$$\lim_{\tau \to 0+} [\varphi_3^\tau(v)](\mathbf{1}) = v(\mathbf{1})$$

and

$$\lim_{\tau \to 0+} ([\varphi_3^\tau(v)](\mathbf{1}+\chi) - [\varphi_3^\tau(v)](\chi)) = v(\mathbf{1}).$$

Therefore,

$$[\varphi_3^L(v)](\mathbf{1}+\chi) = [\varphi_3^L(v)](\chi) + [\varphi_3^L(v)](\mathbf{1})$$

and thus, using (9),

$$[\varphi_3^L(v)](\mathbf{a}+b\chi) = b[\varphi_3^L(v)](\chi) + a[\varphi_3^L(v)](\mathbf{1}).$$

Each of the maps φ_1, φ_2 and φ_3^L is linear, efficient, positive, symmetric, and of norm 1. Moreover, if $v \in Q$, $(\varphi_3^L \circ \varphi_2 \circ \varphi_1)(v) \in Q$. Therefore, there is a value of norm 1 on the space of all games v for which $(\varphi_3^L \circ \varphi_2 \circ \varphi_1)(v) \in Q \cup FA$; $v \mapsto (\varphi_3^L \circ \varphi_2 \circ \varphi_1)(v)$ if $(\varphi_3^L \circ \varphi_2 \circ \varphi_1)(v) \in FA$ and $v \mapsto \varphi[(\varphi_3^L \circ \varphi_2 \circ \varphi_1)(v)]$ if $(\varphi_3^L \circ \varphi_2 \circ \varphi_1)(v) \in Q$. One can use here either our value φ on Q or the value defined in Theorem 2 of [6].

6. Approximation of the value

The formula of $\varphi_\mu^\delta(f \circ \mu)$ is given as an integral of $F_{f,\mu}(\delta, x, S)$ w.r.t. the measure P_μ^δ. The integrand $F_{f,\mu}(\delta, x, S)$ has a tractable expression. On the other hand, the integration measure is known only via its characteristic function which, moreover, depends on the range of μ. In the case that μ is a vector of mutually singular non-atomic measures, or is a linear transformation of a vector of mutually singular non-atomic measures, the integration measure has a much simplified form and can be expressed by means of the classical Cauchy distributions. Indeed, if μ is a vector of mutually singular non-atomic probability measures, the coordinates of a P_μ-distributed random variables are i.i.d. Cauchy random variables.

It is thus of interest to find an approximation of the value φv of a game $v = f \circ (\mu_1, \ldots, \mu_n)$ by a sequence of values φv_k so that, for every fixed k, $v_k = f \circ (\nu_1^k, \ldots, \nu_n^k)$ and each non-atomic measure ν_i^k is a linear combination of a list $\eta_1^k, \ldots, \eta_{m_k}^k$ of mutually singular non-atomic measures.

Let $\mu = (\mu_1, \ldots, \mu_n)$ be an $\mathbb{R}^n$-valued vector of non-atomic measures and $f : \mathcal{R}(\mu) \to \mathbb{R}$ so that $f \circ \mu \in Q$. As mentioned earlier, $\varphi(f \circ \mu)$ is a linear combination of the measures μ_i, $i = 1, \ldots, n$. Therefore

$$\varphi(f \circ \mu) = \sum_{i=1}^{n} a_i(f, \mu)\mu_i.$$

The coefficients $a_i(f, \mu)$ are uniquely defined by φ iff $\mu_1, \ldots, \mu_n$ are linearly independent. The next result comments on the dependence of $a_i(f, \mu)$ on μ.

PROPOSITION 3: *Assume that $\mu = (\mu_1, \ldots, \mu_n)$ and $\nu^k = (\nu_1^k, \ldots, \nu_n^k)$, $k \geq 1$, are vectors of non-atomic probability measures such that $AF(\nu^k) = AF(\mu)$ (from some k on), and $\|y\|_{\nu^k} \to_{k\to\infty} \|y\|_\mu$. Let f be defined on the ranges of μ and ν^k with $f \circ \mu \in Q(\mu)$ and $f \circ \nu^k \in Q(\nu^k)$. Then for every $1 \leq i \leq n$,*

$$a_i(f, \nu^k) \to_{k\to\infty} a_i(f, \mu).$$

Proof: It is sufficient to prove that for every $S \in \mathcal{C}$ and $S_k \in \mathcal{C}$ with $\mu(S) =$

$\nu^k(S_k)$, $k \geq 1$,

$$\varphi v(S) = \lim_{k\to\infty} \varphi v_k(S_k).$$

Note that for every $x \in \mathbb{R}^n$ and δ sufficiently small so that $\delta x \in 2\mathcal{R}(\mu) - \mu(I)$ and $\delta x \in 2\mathcal{R}(\nu^k) - \nu^k(I) = 2\mathcal{R}(\nu^k) - \mu(I)$,

$$F_{f,\mu}(\delta, x, S) = F_{f,\nu^k}(\delta, x, S_k).$$

As $F_{f,\mu}(\delta, x, S)$ is bounded on $\{(\delta, x) : \delta x \in 2\mathcal{R}(\mu) - \mu(I)\}$ and $F_{f,\nu^k}(\delta, x, S_k)$ is bounded on $\{(\delta, x) : \delta x \in 2\mathcal{R}(\nu^k) - \nu^k(I)\}$, it is sufficient to prove that

$$\|P_\mu - P_{\nu^k}\| \to_{k\to\infty} 0,$$

which follows from Lemma 3. ∎

Assume that $(\Pi_k)_{k=1}^\infty$ is an admissible sequence of partitions and that ν is a non-atomic probability measure such that μ_i is absolutely continuous w.r.t. ν. For each k let μ^k be the non-atomic vector measure that coincides with μ on each atom of Π_k and whose Radon–Nikodym derivative w.r.t. ν is constant on each atom of Π_k. It follows that for any increasing chain of coalitions $S_1 \subset \cdots \subset S_m$ there is an increasing sequence of coalitions $T_1 \subset \cdots \subset T_m$ such that $\nu^k(S_i) = \mu(T_i)$ and therefore $f \circ \mu^k \in Q(\mu^k)$ with $\|f \circ \mu^k\| \leq \|f \circ \mu\|$. By Corollary 2, $\|P_\mu - P_{\mu^k}\| \to_{k\to\infty} 0$. Therefore

$$\max_{S:\mu(S)\in\mathcal{R}(\mu^k)} |\varphi(f \circ \mu)(S) - \varphi(f \circ \mu^k)(S)| \to_{k\to\infty} 0.$$

6.1 Formulas for the value. Let $v = f \circ \mu \in Q(\mu)$, where $\mu = (\mu_1, \ldots, \mu_n)$ is a vector of linearly independent non-atomic probability measures and f: $\mathcal{R}(\mu) \to \mathbb{R}$. We will provide equivalent formulas for the value φv in several special cases. It is well known [1] that if f is continuously differentiable on the range of μ,

$$\varphi(f \circ \mu)(S) = \int_0^1 f_{\mu(S)}(t\mu(I))dt,$$

where f_y is the directional derivative of f in the direction y, i.e.,

$$f_y(x) = \lim_{\varepsilon\to 0+} \frac{f(x + \varepsilon y) - f(x)}{\varepsilon}.$$

Many games of interest, e.g., the n-handed glove market, or more generally non-differentiable market games, are not differentiable on the diagonal $[0, \mu(I)]$, and thus the above formula for the value is not applicable to these games.

Our aim here is to approximate $\varphi(f \circ \mu)(S)$ in various special cases as an average of

$$f_{\mu(S)}(t\mu(I) + x),$$

where $0 \leq t \leq 1$ is uniformly distributed on $[0,1]$ and x is distributed on a neighborhood $\mu(\emptyset)$.

A game $v \in BV$ is **absolutely continuous** if there is a non-atomic probability measure ν such that for every $\varepsilon > 0$ there is $\delta > 0$ such that for every increasing sequence of coalitions $S_1 \subset \cdots \subset S_{2k}$ with $\sum_{i=1}^{k} \nu(S_{2i}) - \nu(S_{2i-1}) < \delta$, $\sum_{i=1}^{k} |v(S_{2i}) - v(S_{2i-1})| < \varepsilon$. The set of all absolutely continuous games is denoted AC.

LEMMA 10: *Assume that $f \circ \mu \in AC$. Then for every $y \in \mathcal{R}(\mu)$ the directional derivatives f_y exists a.e. (in the relative interior of $\mathcal{R}(\mu)$) and for every sufficiently small $\delta > 0$ and every coalition $S \in \mathcal{C}$,*

$$\psi_\mu^\delta(f \circ \mu, S) = \int\int I_\delta(t) f_{\mu(S)}(t + \delta^2 x)\, dt\, dP_\mu^\delta(x)$$

is well-defined and

$$|\psi^\delta(f \circ \mu, S) - \varphi_\mu^\delta(f \circ \mu, S)| \to_{\delta \to 0} 0.$$

Proof: Assume $f \circ \mu \in AC$. Then for every x in the relative interior of $\mathcal{R}(\mu)$ and $y \in \mathcal{R}(\mu)$, there is $\alpha > 0$ sufficiently small so that the function $s\colon [0,\alpha] \to \mathbb{R}$ defined by $s(a) = f(x + ay)$ is absolutely continuous and therefore differentiable a.e. on $[0,\alpha]$, implying that for a.e. $a \in [0,\alpha]$, $f_y(x + ay)$ exists and thus, by Fubini's theorem, f_y exists almost everywhere in the relative interior of $\mathcal{R}(\mu)$. Moreover,

$$\frac{f(x+\alpha y) - f(x)}{\alpha} = \frac{1}{\alpha}\int_0^\alpha f_y(x + ay)da = \int_0^1 f_y(x + s\alpha y)ds. \tag{10}$$

Therefore,

$$F_{f,\mu}(\delta, x, S) = \int I_\delta(t) \int_0^1 f_{\mu(S)}(t\mu(I) + \delta^2 x + s\delta^3\mu(S))ds\, dt$$

whenever $\delta x \in 2\mathcal{R}(\mu) - \mu(I)$, and thus

$$\varphi_\mu^\delta(f \circ \mu, S) = \int\int I_\delta(t) \int_0^1 f_{\mu(S)}(t\mu(I) + \delta^2 x + s\delta^3\mu(S))\, ds\, dt\, dP_\mu^\delta. \tag{11}$$

Given $\delta > 0$ sufficiently small, $3\delta < t < 1-3\delta$, $x \in 2\mathcal{R}(\mu) - \mu(I)$, and $S \in \mathcal{C}$, we define

$$\bar{H}(\delta, x, S) = \frac{1}{\delta}\int_0^1 I_\delta(t)\int_0^1 |f_{\mu(S)}(t\mu(I) + \delta x + s\delta\mu(S))|ds\, dt,$$
$$\bar{G}(\delta, x, S) = \bar{H}(\delta, x+\mu(S), S^c).$$

Note that as $f\circ\mu \in AC$, the function $s \mapsto f(t\mu(I) + \delta x + s\delta\mu(S))$ is absolutely continuous in $0 \le s \le 1$ and its variation over the interval $0 \le s \le 1$ equals $\int_0^1 |f_{\mu(S)}(t\mu(I) + \delta x + s\delta\mu(S))|ds$. As in the proof of Lemma 4, one proves, by adding to the chains of ideal coalitions h_k^α, $0 \le k \le 2K(\delta)$, additional elements $h_{k,m}^\alpha = s_{k,m}h_{k+1}^\alpha + (1-s_{k,m})h_{k+1}^\alpha$, $1 \le m \le n_k$, where $0 < s_{k,i} < s_{k,i+1} < 1$, and so that the variation of v over the sequence $h_k^\alpha \le h_{k,1}^\alpha \le \cdots \le h_{k,n_k}^\alpha \le h_{k+1}^\alpha$ is within $\varepsilon > 0$ of $\int_0^1 |f_{\mu(S)}((3\delta + k\delta + \alpha)\mu(I) + \delta x + s\delta\mu(S))|ds$ if k is even and within $\varepsilon > 0$ of $\int_0^1 |f_{\mu(S)}((3\delta + k\delta+\alpha)\mu(I) + \delta x + \delta\mu(S) + s\delta\mu(S^c))|ds$ if k is odd, that

$$\|f\circ\mu\| \ge |\bar{H}(\delta, x, S)| + |\bar{G}(\delta, x, S)|.$$

Therefore, for $\delta > 0$ sufficiently small, for every $x \in \mathbb{R}^n$ with $\delta x \in 2\mathcal{R}(\mu) - \mu(I)$ and $S \in \mathcal{C}$,

$$\int I_\delta(t)\int_0^1 |f_{\mu(S)}(t\mu(I) + \delta^2 x + s\delta^3\mu(S))|dsdt$$

is bounded by $\|f\circ\mu\|$. Applying Fubini's theorem to (11) we deduce that

$$\begin{aligned}\varphi_\mu^\delta(f\circ\mu, S) &= \int_0^1\int\int I_\delta(t)f_{\mu(S)}(t\mu(I) + \delta^2 x + s\delta^3\mu(S))dtdP_\mu^\delta(x)ds\\ &= \int_0^1\int\int I_\delta(t)f_{\mu(S)}(t\mu(I) + \delta^2 x)dtd(P_\mu^\delta * \delta_{-s\delta^3\mu(S)})(x)ds.\end{aligned}$$

As $\sup_{0\le s\le 1}\|P_\mu^\delta - P_\mu^\delta * \delta_{-s\delta^3\mu(S)}\| \to 0$ as $\delta \to 0+$ by Corollary 1,

$$\varphi_\mu^\delta(f\circ\mu, S) - \int\int I_\delta(t)f_{\mu(S)}(t\mu(I) + \delta^2 x)dtdP_\mu^\delta(x) \to_{\delta\to 0+} 0. \qquad \blacksquare$$

Assume that f is concave and homogeneous of degree 1. Then the core of $v = f\circ\mu$, denoted $C(v)$, is a convex compact subset of the linear subspace generated by $\mu_1, \ldots, \mu_n$. Given $x \in \mathbb{R}^n$, we denote by $p(x)$ the set of all elements $\nu \in C(v)$ that minimize $\nu(T)$ where $T \in \mathcal{C}$ is a coalition such that, for some $\eta > 0$, $2\mu(T) - \mu(I) = \eta x$. For almost all x in $\mathbb{R}^n$, $p(x)$ is a singleton. The next proposition shows that our value φ and the Mertens value ψ of the market game $f\circ\mu$ coincide, by demonstrating a formula for $\varphi(f\circ\mu)$ that coincides with the formula for the Mertens value $\psi(f \circ \mu)$ given in [7]. The formula is an analog of the one for the measure-based values given in [3]

PROPOSITION 4: *Assume that f is concave and homogeneous of degree 1. Then the core of $v = f \circ \mu$, $C(v)$, is a convex compact subset of the linear subspace generated by $\mu_1, \ldots, \mu_n$, and φv is given by*

$$\varphi v(S) = \int p(x)(S) dP_\mu(x).$$

Proof: Assume that $S \in \mathcal{C}$, $\delta > 0$ sufficiently small, $3\delta < t < 1 - 3\delta$, and $x \in \mathbb{R}^n$ with $\delta x \in 2\mathcal{R}(\mu) - \mu(I)$. Then $f_{\mu(s)}(t + \delta x) \geq f(\mu(S))$ by concavity and superadditivity of f. By Theorem 24.6 of [12], given $0 < t < 1$ and $x \in \mathbb{R}^n$ such that $p(x)$ is a singleton,

$$\liminf_{\delta \to 0+} f_{\mu(S)}(t + \delta^2 x) \geq p(x)(S).$$

Therefore, using Lemma (10) and Fatou's lemma,

$$\liminf_{\delta \to 0+} \psi_\mu^\delta(f \circ \mu, S) \geq \int p(x)(S) dP_\mu(x).$$

As this holds for every coalition $S \in \mathcal{C}$, and $p(x)(S) + p(x)(S^c) = p(x)(I) = f(\mu(I))$, the equality follows. ∎

The next proposition provides a formula for the value of a game of the form $f \circ \mu$, where f is concave but not necessarily homogeneous of degree 1. Assume that f is concave on the range of a vector of linearly independent non-atomic probability measures $\mu = (\mu_1, \ldots, \mu_n)$. Given $0 < t < 1$ we denote by $A(t)$ the set of all supergradients of f at $t\mu(I)$, i.e., the set of all vectors $a(t,x) \in \mathbb{R}^n$ such that $\langle a(t,x), y - t\mu(I)\rangle \geq f(y) - f(t\mu(I))$ for every $y \in \mathcal{R}(\mu)$. First note that for almost every pair t, x with $0 < t < 1$ and $x \in \mathbb{R}^n$ there is a unique $a(t,x) \in A(t)$ that minimizes $\langle a, x\rangle$. Denote by $p(t,x)$ the non-atomic vector measure $\sum_{i=1}^n a_i(t,x)\mu_i$.

PROPOSITION 5: *Assume that f is concave on the range of a vector of linearly independent non-atomic probability measures $\mu = (\mu_1, \ldots, \mu_n)$ continuous at $\mu(I)$ and $\mu(\emptyset)$. Then the value of $v = f \circ \mu$, φv, is given by*

$$\varphi v(S) = \int \int p(t,x)(S) \, dt \, dP_\mu(x).$$

Proof: Assume that $S \in \mathcal{C}$, $\delta > 0$ sufficiently small, $3\delta < t < 1 - 3\delta$, and $x \in \mathbb{R}^n$ with $\delta x \in 2\mathcal{R}(\mu) - \mu(I)$. By Theorem 24.6 of [12], given $0 < t < 1$ and $x \in \mathbb{R}^n$

such that $p(t,x)$ is a singleton,

$$\liminf_{\delta\to 0+} f_{\mu(S)}(t+\delta^2 x) \geq p(t,x)(S).$$

As f is concave, it is Lipschitz on a neighborhood of the interval $[\varepsilon\mu(I),(1-\varepsilon)\mu(I)]$. Given $\eta > 0$, there is $\varepsilon > 0$ sufficiently small such that, for $\delta > 0$ sufficiently small,

$$\psi_\mu^\delta(f\circ\mu, S) \geq \int\int_\varepsilon^{1-\varepsilon} I_\delta(t) f_{\mu(S)}(t+\delta^2 x)\, dt\, dP_\mu^\delta(x) - \eta$$

and

$$\int\int p(t,x)(S)dtdP_\mu(x) \geq \int\int_\varepsilon^{1-\varepsilon} p(t,x)(S)dtdP_\mu(x) - \eta.$$

Therefore, using Lemma (10) and Fatou's lemma,

$$\liminf_{\delta\to 0+} \psi_\mu^\delta(f\circ\mu, S) \geq \int\int p(t,x)(S)dtdP_\mu(x) - 2\eta.$$

As this holds for every $\eta > 0$,

$$\liminf_{\delta\to 0+} \psi_\mu^\delta(f\circ\mu, S) \geq \int\int p(t,x)(S)dtdP_\mu(x)$$

and therefore

$$\varphi(f\circ\mu)(S) \geq \int\int p(t,x)(S)dtdP_\mu(x). \tag{12}$$

It remains to prove that $\varphi(f\circ\mu)(S) \leq \int\int p(t,x)(S)dtdP_\mu(x)$. Note that $t\mapsto f(t\mu(I))$ is a concave function and therefore differentiable a.e. and therefore for a.e. $0<t<1$ and $x\in\mathbb{R}^n$, we have $p(t,x)(I) = \frac{d}{dt}f(t\mu(I))$ and thus, for every x, we have $\int p(t,x)(I)\,dt = f(\mu(I)) = \varphi(f\circ\mu)(I)$. As $p(t,x)(S)+p(t,x)(S^c) = p(t,x)(I)$,

$$\begin{aligned}\varphi(f\circ\mu)(I) =& \varphi(f\circ\mu)(S) + \varphi(f\circ\mu)(S^c)\\ \geq& \int\int [p(t,x)(S)+p(t,x)(S^c)]dtdP_\mu(x)\\ =& \varphi(f\circ\mu)(I)\end{aligned}$$

and thus all the weak inequalities are equalities. Together with Inequality (12), applied to S^c, we deduce that $\varphi(f\circ\mu)(S) \leq \int\int p(t,x)(S)dtdP_\mu(x)$. ∎

References

[1] R. J. Aumann and L. S. Shapley, *Values of Non-Atomic Games*, Princeton University Press, Princeton, NJ, 1974.

[2] A. Dvoretzky, A. Wald and J. Wolfowitz, *Relations among certain ranges of vector measures*, Pacific Journal of Mathematics **1** (1951), 59–74.

[3] S. Hart, *Measure-based values of market games*, Mathematics of Operations Research **5** (1980), 192–228.

[4] S. Hart and A. Neyman, *Values of vector measure games: Are they linear combinations of the measures?*, Journal of Mathematical Economics **17** (1988), 31–40.

[5] A. Lyapunov, *Sur les fonctions-vecteurs completement additives*, Bulletin of the Academy of Sciences of the USSR, Mathematics Series **4** (1940), 465–478.

[6] J.-F. Mertens, *The Shapley value in the non-differentiable case*, International Journal of Game Theory **17** (1988), 1–65.

[7] J.-F. Mertens, *Nondifferentiable TU markets: The value*, in *The Shapley Value* (A. Roth, ed.), Cambridge University Press, 1988, pp. 235–264.

[8] D. Monderer and A. Neyman, *Values of smooth non-atomic games: The method of multilinear approximation*, in *The Shapley Value* (A. Roth, ed.), Cambridge University Press, 1988, pp. 217–234.

[9] A. Neyman, *Continuous values are diagonal*, Mathematics of Operations Research **2** (1977), 338–342.

[10] A. Neyman, *Weighted majority games have asymptotic value*, Mathematics of Operations Research **13** (1988), 556–580.

[11] A. Neyman and Y. Tauman, *The existence of nondiagonal axiomatic values*, Mathematics of Operations Research **1** (1976), 246–250.

[12] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.